EXAMPLES OF TOPOLOGICAL SPACES WITH ARBITRARY COHOMOLOGY JUMP LOCI

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Abstract. Given any subvariety of a complex torus defined over $\mathbb{Z}$ and any positive integer k, we construct a finite CW complex X such that the k-th cohomology jump locus of X is equal to the chosen subvariety, and the i-th cohomology jump loci of X are trivial for $i < k$.

1. Introduction

The motivation of this note is to explore the consequence of the results of [3] and [1], that the cohomology jump loci of a smooth complex quasi-projective variety are unions of torsion translates of subtori.

Given a topological space M, we denote by $L(M)$ the space of rank one local systems on M, which is equal to $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ and has a group structure. When $\pi_1(M)$ is finitely generated, $L(M)$ has a complex variety structure which is isomorphic to the cartesian product of $(\mathbb{C}^*)^{b_1(M)}$ and a discrete finite abelian group. In $L(M)$, there are some canonically defined subvarieties called the cohomology jump loci (or characteristic varieties). They are defined to be $\Sigma^i_r(M) = \{\rho \in L(M) \mid \dim H^i(M, L_\rho) \geq r\}$, where L_ρ is the rank one local system on M associated to the representation ρ. They are always varieties defined over $\mathbb{Z}$. When $r = 1$, we omit r and just write $\Sigma^i(M)$.

The main result of this note is the following.

Theorem 1.1. *Fix a positive integer n. Let Z be any (not necessarily irreducible) subvariety of $(\mathbb{C}^*)^n$, which is defined over $\mathbb{Z}$, and let k be any positive integer. There exists a finite CW complex M, such that $L(M) = (\mathbb{C}^*)^n$, $\Sigma^k(M) \cup \{\mathbb{1}\} = Z \cup \{\mathbb{1}\}$, and $\Sigma^i(M)$ is either empty or equal to $\{\mathbb{1}\}$ for $i < k$.*

We prove the theorem in the rest of this note. The proof is divided into two parts, $k = 1$ and $k \geq 2$. When $k = 1$, this is a group theoretic problem, and is essentially known (for example, [5] Lemma 10.3). For completeness, we include the proof still. In fact, the examples for the case $k \geq 2$ are inspired by the case $k = 1$.

When $k \geq 2$ and n is even, and when Z is not a union of torsion translates of subtori, our examples are homotopy $(k-1)$-equivalent to an abelian variety, but not homotopy k-equivalent to any quasi-projective variety. This shows that the result of [1] puts genuine higher homotopy obstruction to the possible homotopy types of quasi-projective varieties.

Carlos Simpson has kindly pointed out to us that the construction for $k \geq 2$ has appeared in [4]. Thus, this note is a self contained proof of some essentially known result.

2. Constructing the examples for $k \geq 2$

We assume $k \geq 2$ though out this section.

Under the notation of Theorem 1.1, we start with a real torus $M_0 = (S^1)^n$. Let N_0 be the universal cover of M_0, and let the covering map be $p_0 : N_0 \to M_0$. Fixing an origin $O \in M_0$, we attach a k-sphere S^k to M_0 at O, obtaining a new space M_1. In other words, M_1 is the wedge sum of M_0 and S^k. We call such a k-sphere. Let $p_1 : N_1 \to M_1$ be the universal covering map. Then N_1 is obtained from N_0 by attaching infinitely many k-spheres parametrized by $\mathbb{Z}^n$. Denote the k-sphere in N_1 corresponding to $J \in \mathbb{Z}^n$ by B_J.

Suppose the subvariety Z of $(\mathbb{C}^*)^n$ is defined by Laurent polynomials $f_i(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$, for $1 \leq i \leq r$, where x_j's are the coordinates of $(\mathbb{C}^*)^n$. Suppose $f_i(x_1, \ldots, x_n) = \sum_{J \in \Lambda_i} a_i^J x^J$, where Λ_i is a finite subset of $\mathbb{Z}^n$, a_i^J are integers and x^J is the product of x_j's under the usual multi-index notation. For example, if $n = 2$ and $J = (-2, 3)$, then $x^J = x_1^{-2} x_2^3$.

The CW complex M satisfying the property of Theorem 1.1 will be obtained by gluing r $(k+1)$-cells to M_1 corresponding to the Laurent polynomials $f_1, \ldots, f_r$. To explain how to attach the $(k+1)$-cell corresponding to one Laurent polynomial f_i, it is easier to work on N_1. Fix a $J_0 \in \mathbb{Z}^n$. We can attach a $(k+1)$-cell $e_{J_0}^{k+1}$ to N_1, such that $\partial e_{J_0}^{k+1}$ represents the cycle $\sum_{J \in \Lambda_1} a_i^J B_{J_0 + J}$ in $H_k(N_1, \mathbb{Z})$. Denote the new space by N_{2,J_0}. Notice that N_1 is $(k-1)$-connected. Hence N_{2,J_0} is uniquely determined up to homotopy. Define N_2 to be the coproduct of N_{2,J_0} over N_1 for all $J_0 \in \mathbb{Z}^n$. Then N_2 is obtained from N_1 by attaching infinitely many $(k+1)$-cells parametrized by $\mathbb{Z}^n$. Suppose we attach these $(k+1)$-cells in a compatible way. Then the Galois action on N_1 by $\mathbb{Z}^n$ extends to N_2. Now, let M_2 be the quotient of N_2 by the Galois action $\mathbb{Z}^n$. By our construction, M_2 is obtained by attaching one $(k+1)$-cell to M_1. In the same way, we can attach more $(k+1)$-cells corresponding to other Laurent polynomials $f_2, \ldots, f_r$. Let M be the space we obtained this way, and let N be the cover of M with Galois group $\mathbb{Z}^n$. Denote the covering map by $p : M \to N$.

By our construction, there is a natural isomorphism $\pi_1(M) \cong \pi_1(M_0)$. Since $M_0 = (S^1)^n$, there is a natural isomorphism $L(M) \cong (\mathbb{C}^*)^n$, and the isomorphism is induced by an isomorphism of the underlying scheme over $\mathbb{Z}$.

Proposition 2.1. *Under the above isomorphism, we have the following.*

(1) $\Sigma^i(M) = \{\mathbb{1}\}$ *for* $0 \leq i \leq \min\{n, k-1\}$, $\Sigma^i(M) = \emptyset$ *for* $n < i \leq k-1$.
(2) $\Sigma^k(M) = Z \cup \{\mathbb{1}\}$ *when* $k \leq n$, *and* $\Sigma^k(M) = Z$ *when* $k > n$.
(3) M *is homotopy* $(k-1)$*-equivalent to the real torus* $(S^1)^n$.
(4) M *is not homotopy* k*-equivalent to any quasi-projective variety.*

The statements (1) and (3) are obvious from the construction. Moreover, (4) follows from (2) and [1]. To prove this, one has to argue that the cohomology jump locus Σ^i is a homotopy i-equivalence invariant. We leave this to the reader. We will prove statement (2) in the rest of this section.

We will compute the cohomology jump loci of M via Alexander modules. Recall that $p : N \to M$ is the covering map with Galois action by $\mathbb{Z}^n$. Then

in a natural way, the homology groups $H_i(N,\mathbb{Z})$ become $\mathbb{Z}^n$-modules. Notice that $\mathbb{Z}^n$ is naturally identified with $H_1(M,\mathbb{Z})$, and in a natural way $L(M) = \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}^*) \cong (\mathbb{C}^*)^n$. Thus, the group ring of $\mathbb{Z}^n$ is naturally identified with $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$, which is the coordinate ring of the underlying $\mathbb{Z}$-scheme of $(\mathbb{C}^*)^n$. As $\mathbb{Z}^n$-module, $H_i(N,\mathbb{Z})$ has now a natural $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$-module structure. These $H_i(N,\mathbb{Z})$, together with the $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$-module structures are called Alexander modules.

Denote the support of the Alexander modules $H_i(N,\mathbb{Z})$ in $(\mathbb{C}^*)^n$ by $\mathcal{V}^i(M)$. Then the cohomology jump loci and the support of the Alexander modules are closely related by the following theorem of Papadima and Suciu.

Theorem 2.2 ([2] Theorem 3.6)**.**

$$\bigcup_{i=0}^{l}\Sigma^i(M) = \bigcup_{i=0}^{l}\mathcal{V}^i(M)$$

for any integer $l \geq 0$.

Since M is homotopy $(k-1)$-equivalent to the real torus $(S^1)^n$, N is homotopy $(k-1)$-equivalent to a point. Therefore, $\mathcal{V}^0(M) = \{\mathbb{1}\}$ and $\mathcal{V}^i(M) = \emptyset$ for $1 \leq i \leq k-1$. According to Theorem 2.2, $\Sigma^i(M) \subset \{\mathbb{1}\}$.

By construction, $H_k(N_1,\mathbb{Z})$ is a free $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$-module of rank one. Recall that N is obtained from N_1 by attaching many $(k+1)$-cells. Each $(k+1)$ gives a relation in $H_k(N_1,\mathbb{Z})$ as $\mathbb{Z}$-module. After attaching infinitely many cells, $H_k(N_2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]/(f_1)$ as $\mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$-modules. This can be proved by a standard Mayer-Vietoris sequence argument. We leave this to the reader. Similarly, $H_k(N,\mathbb{Z}) \cong \mathbb{Z}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]/(f_1,\dots,f_r)$. Thus, $\mathcal{V}^k(M) = Z$. Now, according to Theorem 2.2, $\Sigma^k(M) \cup \{\mathbb{1}\} = Z \cup \{\mathbb{1}\}$.

To check whether $\mathbb{1} \in \Sigma^k(M)$ is to compute whether $H^k(M,\mathbb{C}) = 0$. This can be done directly from the construction. In fact, $H^k(M,\mathbb{C}) \neq 0$ when $k \leq n$ and $H^k(M,\mathbb{C}) = 0$ when $k > n$. Thus we have proved the proposition.

Remark 2.3. One can try to use the same construction for $k = 1$. It works except that the circle (or 1-shpere) we attach may create extra elements in $H_1(M,\mathbb{Z})$. In fact, it works when Z does not contain any torsion point.

3. Group theoretic first cohomology jump loci

It is an easy and well known fact that for a topological space M and $\rho \in L(M)$, $H^1(M,L_\rho) \cong H^1(\pi_1(M),\mathbb{C}_\rho)$. Here the representation $\rho : \pi_1(M) \to \mathbb{C}^*$ gives $\mathbb{C}$ a $\pi_1(M)$-module structure, and we emphasize the $\pi_1(M)$-module structure on $\mathbb{C}$ by writing $\mathbb{C}_\rho$. Therefore, the first cohomology jump loci of M only depend on $\pi_1(M)$.

Definition 3.1. *Let G be a finitely presented group. We denote the character variety* $\mathrm{Hom}(G,\mathbb{C}^*)$ *by $L(G)$, and define the group theoretic cohomology jump loci* $\Sigma^i(G) = \{\rho \in L(G) \mid H^i(G,\mathbb{C}_\rho) \neq 0\}$.

The isomorphism between the cohomology of local system and the group cohomology shows that the natural isomorphism $L(M) \cong L(\pi_1(M))$ induces an isomorphism between subvarieties $\Sigma^1(M) \cong \Sigma^1(\pi_1(M))$. Given a finitely presented group G, there is a standard process to construct a finite CW complex with fundamental group G. Therefore, the case $k = 1$ of Theorem 1.1 is equivalent to the following.

Proposition 3.2. *Let Z be any (not necessarily irreducible) subvariety of $(\mathbb{C}^*)^n$ defined over $\mathbb{Z}$. There is a finitely presented group G such that $L(G) = (\mathbb{C}^*)^n$ and $\Sigma^1(G) = Z \cup \{\mathbb{1}\}$.*

Before proving the Proposition, we give an algorithm to compute the first group theoretic cohomology jump loci, which is slightly different from [5].

$H^1(G, \mathbb{C}_\rho)$ can be computed by the quotient of 1-cycles by 1-boundaries, i.e., $H^1(G, \mathbb{C}_\rho) = Z^1(G, \mathbb{C}_\rho)/B^1(G, \mathbb{C}_\rho)$, where

$$Z^1(G, \mathbb{C}_\rho) = \{\tau \in \mathrm{Hom}_{set}(G, \mathbb{C}) \mid \tau(ab) = \rho(a)\tau(b) + \tau(a) \text{ for any } a, b \in G\}$$

and

$$B^1(G, \mathbb{C}_\rho) = \{\tau \in \mathrm{Hom}_{set}(G, \mathbb{C}) \mid \tau(a) = \rho(a)\tau(1) - \tau(1) \text{for any } a \in G\}.$$

Denote the commutator subgroup of G by G' and denote the abelianization of G by $\mathrm{Ab}(G)$, then a straightforward computation shows the following.

Lemma 3.3. *When $\rho \neq \mathbb{1}$,*

$$H^1(G, \mathbb{C}_\rho) \cong \{\tau \in \mathrm{Hom}(G', \mathbb{C}) \mid \tau(sa) = \rho(s)\tau(a) \text{ for any } s \in G, a \in G'\}.$$

Given a finitely presented group G, we define an element $g \in G$ to be in the "torsion free metabelian kernel", if $g \in G'$, and the image of g in $\mathrm{Ab}(G')$ is torsion. It is very easy to check that the torsion free metabelian kernel forms a normal subgroup. We define the quotient of G by its torsion free metabelian kernel to be the torsion free metabelianization of G, denoted by $\mathrm{TFM}(G)$. The following is a direct consequence of the previous lemma.

Corollary 3.4. *The first cohomology jump locus of G only depends on $\mathrm{TFM}(G)$. More preciously, the natural isomorphism $L(G) \cong L(\mathrm{TFM}(G))$ induces an isomorphism $\Sigma^1(G) \cong \Sigma^1(\mathrm{TFM}(G))$.*

Given a finitely presented group G, we define the first Alexander module of G to be the kernel of the natural map $\mathrm{TFM}(G) \to \mathrm{Ab}(G)$, or equivalently the commutator subgroup of $\mathrm{TFM}(G)$. Denote the first Alexander module of G by $\mathrm{Alex}(G)$. Then we have a short exact sequence of groups,

$$0 \to \mathrm{Alex}(G) \to \mathrm{TFM}(G) \to \mathrm{Ab}(G) \to 0.$$

This short exact sequence induces an action of $\mathrm{Ab}(G)$ on $\mathrm{Alex}(G)$, which gives $\mathrm{Alex}(G)$ a $\mathbb{Z}[\mathrm{Ab}(G)]$-module structure. As an analog of Theorem 2.2, the following follows from Lemma 3.3.

Corollary 3.5. *Suppose $b_1(G) > 0$. The coordinate ring of the underlying $\mathbb{Z}$-scheme of $L(G)$ is naturally isomorphic to $\mathbb{Z}[\mathrm{Ab}(G)]$. Under this isomorphism,*

$$\Sigma^1(G) = \mathrm{Supp}\,(\mathrm{Alex}(G)) \cup \{\mathbb{1}\}.$$

Proof of Proposition 3.2. Now, we are ready to construct the example satisfying the condition in the proposition. As before, we assume the defining equations of Z to be $f_1, \dots, f_r$, where $f_i \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$.

We start with G_0, which we define to be the free group of n generators $g_1, \dots, g_n$. Let G_1 be the torsion free metabelianization of G_0. The choice of the generators of G gives a natural isomorphism $\mathrm{Ab}(G_1) \cong (\mathbb{Z})^n$. Let $x_1, \dots, x_n$ be the natural coordinates on $(\mathbb{Z})^n$. Then $\mathrm{Alex}(G_1)$ has a natural $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module structure. Since G_0 is the fundamental group of n-loops, we can realize $\mathrm{Alex}(G_1) \cong \mathrm{Alex}(G_0)$ as the first homology group of the integral "net" $N \subset \mathbb{R}^n$. Here $N = \bigcup_{1\leq i\leq n}(\mathbb{Z}^n + l_i)$, where l_i is the i-th coordinate axis. $\mathbb{Z}^n$ acts on N by translation. Hence $\mathbb{Z}^n$ also acts on $\mathrm{Alex}(G_1) \cong H_1(N, \mathbb{Z})$. In fact, the $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module structure on $\mathrm{Alex}(G_1)$ can be interpreted this way.

As a $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module, $H_1(N, \mathbb{Z})$ is generated by the unit squares in each coordinate planes, with a proper choice of orientation. Denote these squares by γ_{ij}, $1 \leq a < b \leq n$. More canonically, we will allow $a \geq b$ and put the condition $\gamma_{ab} + \gamma_{ba} = 0$. Each unit cubic induces a relation between these generators. In fact, $H_1(N, \mathbb{Z}) \cong \bigoplus_{1\leq a,b\leq n} / I$, where I is the ideal generated by elements $\gamma_{ab} + \gamma_{ba}$ and $(x_a - 1)\gamma_{bc} + (x_b - 1)\gamma_{ca} + (x_c - 1)\gamma_{ab}$ for all $1 \leq a, b, c \leq n$. It is not hard to see that as a $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module, $\mathrm{Alex}(G_1)$ has rank $n - 1$.

Since $H_1(N, \mathbb{Z})$ is a $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module, for each $1 \leq i \leq r$ and $1 \leq a, b \leq n$, $h^i_{ab} \overset{\text{def}}{=} f_i(x_1, , x_1^{-1}, \dots, x_n, x_n^{-1})\gamma_{ab}$ is a well-defined element in $H_1(N, \mathbb{Z})$. By our construction, the isomorphism $\mathrm{Alex}(G_1) \cong H_1(N, \mathbb{Z})$ sends $g_a g_b g_a^{-1} g_b^{-1}$ to γ_{ab}. Denote the element in $\mathrm{Alex}(G_1)$ corresponding to h^i_{ab} by $\tilde{h}^i_{ab}$. Let H_2 be the normal subgroup of G_2 generated by $\tilde{h}^i_{ab}$, for all $1 \leq i \leq r$ and $1 \leq a, b \leq n$. Now, define G_3 to be the quotient G_2/H_2. Then $\mathrm{Ab}(G_3) = \mathrm{Ab}(G_2)$, and by construction, we have an isomorphism of $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-modules,

$$\mathrm{Alex}(G_3) \cong \mathrm{Alex}(G_2) \otimes_{\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]} \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]/(f_1, \dots, f_r).$$

Since $\mathrm{Alex}(G_2)$ is of rank $n-1$ as a $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$-module, $\mathrm{Supp}\,(\mathrm{Alex}(G_2)) = L(X)$. Therefore, $\mathrm{Supp}\,(\mathrm{Alex}(G_3))$ is the subvariety defined by $(f_1, \dots, f_r)$, that is Z.

Finally, by Corollary 3.5, $\Sigma^1(G_3) = Z \cup \{\mathbb{1}\}$. Thus we have finished the proof of the proposition. ☐

Remark 3.6. For a general connected finite CW-complex X, $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ is a direct product of a complex torus with a finite abelian group. So one can ask a more general question, "for a finite abelian group A, and any (not necessarily irreducible) variety $Z \subset (\mathbb{C}^*)^n \times A$, does Theorem 1.1 hold?" The answer is yes, and we briefly describe how to adjust our previous constructions to this situation.

First, we can write the coordinate ring of $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ as

$$R \overset{\text{def}}{=} \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, y_1, \dots, y_m]/(y_1^{l_1}, \dots, y_m^{l_m}).$$

Suppose Z is defined by ideal $I = (f_1, \dots, f_r)$ of R.

In the case of $k \geq 2$, we start with $M_0 = (S^1)^n \times Y$, where Y is an Eilenberg-Maclane space of type $K(A, 1)$. Attaching a k-sphere to M_0, we obtain M_1. Then,

similar to the previous construction in section 2, we can attach r $(k+1)$-cells to M_1 corresponding to the functions $f_1, \dots, f_r$. Denote the resulting space by M_2. Now, it follows from Theorem 2.2 and the same argument as before that M_2 satisfies the cohomology jump loci property in Theorem 1.1. Notice that Y can be chosen to be a finite-type CW-complex, but it may not be finite. Therefore, even though M_2 may not be a finite CW-complex, it is of finite type. However, by replacing M_2 by its k'-skeleton, where k' is sufficiently large, it serves as the example of Theorem 1.1.

In the case of $k = 1$, again it suffices to talk about group theoretic cohomology jump loci. We start with the group

$$G_0 =< g_1, \dots, g_n, h_1, \dots, h_m \,|\, h_1^{l_1} = 1, \cdots, h_m^{l_m} = 1 > .$$

Let $G_1 = \mathrm{TFM}(G_0)$. Then $\mathrm{Alex}(G_1) = \mathrm{Alex}(G_0)$ is a finitely generated R-module, locally with positive rank. Similar to what we did earlier in this section, by taking a quotient of G_1 by the subgroup corresponding to $I \cdot \mathrm{Alex}(G_1)$, we obtain a group G_2. Then $\mathrm{Alex}(G_2) \cong \mathrm{Alex}(G_1) \otimes_R R/I$. According to Corollary 3.5, we have

$$\Sigma^1(G_2) = Z \cup \{\mathbb{1}\}.$$

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